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Response of Self-Excited Three-Degree-of-Freedom Systems to Multifrequency Excitations

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The response to multifrequency excitation of a three-degree-of-freedom self-excited system is analyzed by using multiple scales. Five cases of resonance are considered: Harmonic, subharmonic, superharmonic, simultaneous sub/super-harmonic, and combination resonances. The steady-state amplitudes for each case are plotted, showing the influences of the several parameters. Approximate solutions are found and stability analyses are carried out for each case.

1. INTRODUCTION

Considerable attention has been given to the response of oscillatory systems (Krylov and Bogoliubov, 1947; Minorsky, 1962; Haag, 1962; Elnaggar, 1976). The response of these systems is of considerable interest to several fields of engineering, statistical mechanics, electronics, and biological science (Pavlidis, 1962; Lindsay, 1972). The subject has by no means been exhausted, and little attention has been given to the response of oscillatory systems with many degrees of freedom (Tso and Asmis, 1974; Nayfeh and Mook, 1979; Nayfeh, 1983; Nayfeh and Zavodney, 1986). This paper extends the analysis developed in Elnaggar and El-Bassiouny (1991) to a self-excited system with three degrees of freedom in which the first and the second modes are excited externally to multifrequency excitations and the third mode is excited through coupling. The mathematical model of these systems is

$$\ddot{X}_j+\omega_j^2X_j=\varepsilon\left(\dot{X}_j-\frac{1}{3}\dot{X}_j^3\right)+\delta_{j1}\left(\varepsilon\alpha_1X_2+\varepsilon\alpha_2X_3+\sum_{n=1}^{N}F_n\cos(\Omega_nt+\nu_n)\right)$$
$$+\delta_{j2}\left(\varepsilon\alpha_2X_1+\varepsilon\alpha_4X_3+\sum_{m=1}^{M}H_m\cos(\Phi_mt+\gamma_m)\right)$$
$$+\delta_{j3}(\varepsilon\alpha_5X_1+\varepsilon\alpha_6X_2) \qquad (1.1, 1.2, 1.3)$$

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where Ω_n, Φ_m, ν_n, γ_m, F_n, H_m, ω_j, and α_i $(i=1,\ldots,6)$ are constants.

In equations (1.1)-(1.3), and in all the subsequent equations, the subscript j takes the values $j=1,2,3$ and δ_{jk} is the Kronecker symbol, with $k=1,2,3$. This model represents the oscillation of a cutting tool, the self-rotation of elastic rotors, and many other physical systems (Vasilenko, 1969; Kononenko and Kovalchak, 1973).

2. ANALYSIS

Using the method of multiple scales (Nayfeh and Mook, 1979), one seeks an approximate solution in the form

$$X_j(t;\varepsilon)\simeq X_{j0}(T_0,T_1)+\varepsilon X_{j1}(T_0,T_1)+\cdots \qquad (2.1, 2.2, 2.3)$$

Substituting (2.1)-(2.3) into equations (1.1)-(1.3), expanding the derivatives, and equating the coefficients of ε^0 and ε on both sides, we obtain

$$D_0^2X_{j0}+\omega_j^2X_{j0}=\delta_{j1}\sum_{n=1}^{N}K_n\cos(\Omega_nT_0+\nu_n)$$
$$+\delta_{j2}\sum_{m=1}^{M}H_m\cos(\Phi_mT_0+\gamma_m) \qquad (2.4, 2.5, 2.6)$$

$$D^2X_{j1}+\omega_j^2X_{j1}=-2D_0D_1X_{j0}+D_0X_{j0}-\tfrac{1}{3}(D_0X_{j0})^3+\delta_{j1}(\alpha_1X_{20}+\alpha_3X_{30})$$
$$+\alpha_{j2}(\alpha_2X_{10}-\alpha_4X_{30})+\delta_{j3}(\alpha_5X_{10}+\alpha_6X_{20}) \qquad (2.7, 2.8, 2.9)$$

The solutions (2.4)-(2.6) can be expressed as

$$X_{j0}=A_j(T_1)\,e^{i\omega_jT_0}+\delta_{j1}\sum_{m=1}^{N}P_n\Omega_n^{-1}\,e^{i\Omega_nT_0}$$
$$+\delta_{j2}\sum_{m=1}^{M}P_m\Phi_m^{-1}\,e^{i\phi_LT_0}+\text{C.C.} \qquad (2.10, 2.11, 2.12)$$

where

$$P_n=\tfrac{1}{2}F_n\Omega_n(\omega_1^2-\Omega_n^2)\,e^{i\nu_n}, \qquad P_m=\tfrac{1}{2}H_m(\omega_2^2-\Phi_m^2)\,e^{i\gamma_m} \qquad (2.13)$$

where C.C. stands for the complex conjugate of the preceding term. Substituting expressions (2.10)-(2.12) into equations (2.7)-(2.9) yields

$$D_0^2X_{j1}+\omega_j^2X_{j1}$$
$$=i\omega_j(A_j-2A_j'-A_j^2\bar{A}_j\omega_j^2)\,e^{i\omega_jT_0}+\delta_{j1}\Bigg[i\sum_{n=1}^{N}P_n\,e^{i\Omega_nT_0}$$

$$+\alpha_1\left(A_2\,e^{i\omega_2 T_0}+\sum_{m=2}^{M} P_m\Phi_m^{-1}\,e^{i\Phi_m T_0}\right)+\alpha_3 A_3\,e^{i\omega_3 T_0}$$

$$+\frac{1}{3}\,i\left(A_1^3\omega_1^3\,e^{3i\omega_1 T_0}+3A_1^2\omega_1^2\,e^{2i\omega_1 T_0}\sum_{n=1}^{N} P_n\,e^{i\Omega_n T_0}\right.$$

$$-3A_1^2\omega_1^2\,e^{2i\omega_1 T_0}\sum_{n=1}^{N}\bar{P}_n\,\bar{e}^{i\Omega_n T_0}$$

$$+3A_1\omega_1\,e^{i\omega_1 T_0}\sum_{n=1}^{N}\sum_{p=1}^{N} P_nP_p\,e^{i(\Omega_n+\Omega_p)T_0}$$

$$-6A_1\omega_1\,e^{i\omega_1 T_0}\sum_{n=1}^{N}\sum_{p=1}^{N} P_n\bar{P}_p\,e^{i(\Omega_n-\Omega_p)T_0}$$

$$-3\bar{A}_1\omega_1\,\bar{e}^{i\omega_1 T_0}\sum_{n=1}^{N}\sum_{p=1}^{N} P_nP_p\,e^{i(\Omega_n+\Omega_p)T_0}$$

$$-6A_1\bar{A}_1\omega_1^2\sum_{n=1}^{N} P_n\,e^{i\Omega_n T_0}+\sum_{n=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N} P_nP_pP_q\,e^{i(\Omega_n+\Omega_p+\Omega_q)T_0}$$

$$\left.\left.-3\sum_{n=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N} P_nP_p\bar{P}_q\,e^{i(\Omega_n+\Omega_p-\Omega_q)}\right)\right]$$

$$+\delta_{j2}\left[i_m\sum_{m=1}^{M} P_m\,e^{i\Phi_m T_0}+\alpha_2\left(A_1\,e^{i\omega_1 T_0}+\sum_{n=1}^{N} P_n\Omega_n^{-1}\,e^{i\Omega_n T_0}\right)\right.$$

$$+\alpha_4 A_3\,e^{i\omega_3 T_0}+\frac{1}{3}\,i\left(A_2^3\omega_2^3\,e^{3i\omega_2 T_0}+3A_2\omega_2\,e^{2i\omega_2 T_0}\sum_{m=1}^{M} P_m\,e^{i\Phi_m T_0}\right.$$

$$-3A_2^2\omega_2^2\,e^{2i\omega_2 T_0}\sum_{m=1}^{M}\bar{P}_m\,e^{-i\Phi_m T_0}$$

$$+3A_2\omega_2\,e^{i\omega_2 T_0}\sum_{m=1}^{M}\sum_{p=1}^{M} P_mP_p\,e^{i(\Phi_m+\Phi_p)T_0}$$

$$-6A_2\omega_2\,e^{i\omega_2 T_0}\sum_{m=1}^{M}\sum_{p=1}^{M} P_mP_p\,e^{i(\Phi_m-\Phi_p)T_0}$$

$$-3A_2\omega_2\,e^{i\omega_2 T_0}\sum_{m=1}^{M}\sum_{p=1}^{M} P_mP_p\,e^{i(\Phi_m+\Phi_p)T_0}$$

$$-6A_2\bar{A}_2\omega_2^2\sum_{m=1}^{M} P_m\,e^{i\Phi_m T_0}+\sum_{m=1}^{M}\sum_{p=1}^{M}\sum_{q=1}^{M} P_mP_pP_q\,e^{i(\Phi_m+\Phi_p+\Phi_q)T_0}$$

$$-3\sum_{m=1}^{M}\sum_{p=1}^{M}\sum_{q=1}^{M}P_mP_p\bar{P}_q\,e^{i(\Phi_m+\Phi_p-\Phi_q)T_0}\Big)\Big]$$

$$+\delta_{j3}\Big[\alpha_5\Big(A_1\,e^{i\omega_1 T_0}+\sum_{n=1}^{N}P_n\Omega_n^{-1}\,e^{i\Omega_n T_0}\Big)$$

$$+\alpha_6\Big(A_2\,e^{i\omega_2 T_0}+\sum_{m=1}^{M}P_m\Phi_m^{-1}\,e^{i\Phi_m T_0}\Big)$$

$$+\frac{1}{3}\,iA_3^3\omega_3^3\,e^{3i\omega_3 T_0}\Big]+\text{C.C.} \qquad (2.14, 2.15, 2.16)$$

Any particular solution of equations (2.14)-(2.16) contains secular terms and it may contain small divisor terms, depending on the resonances produced in the different cases. To express the nearness of ω_2 to ω_1 and of ω_3 to ω_1, we can write

$$\omega_2=\omega_1+\varepsilon\sigma_1 \qquad \text{and} \qquad \omega_3=\omega_1+\varepsilon\sigma_2 \qquad (2.17)$$

3. HARMONIC RESONANCE

3.1. Approximate Solutions

In this case, we take $F_1=2\varepsilon f_1$ and $H_1=2\varepsilon h_1$. Equations (2.14)-(2.16) become

$$D_0^2X_{j1}+\omega_j^2X_{j1}$$

$$=i\omega_j(A_j-2A_j'-A_j^2\bar{A}_j\omega_j^2)\,e^{i\omega_j T_0}$$

$$+\delta_{j1}\Big[\alpha_1A_2\,e^{i\omega_2 T_0}+\alpha_3A_3\,e^{i\omega_3 T_0}$$

$$-2iA_1\omega_1\,e^{i\omega_1 T_0}\sum_{n=2}^{N}\sum_{p=2}^{N}P_n\bar{P}_p\,e^{i(\Omega_n-\Omega_p)T_0}+2f_1\cos(\Omega_1 T_p+\nu_1)\Big]$$

$$+\delta_{j2}\Big[\alpha_2A_1\,e^{i\omega_1 T_0}+\alpha_4A_3\,e^{i\omega_3 T_0}$$

$$-2iA_2\omega_2\,e^{i\omega_2 T_0}\sum_{m=2}^{M}\sum_{p=2}^{M}P_m\bar{P}_p\,e^{i(\Phi_m-\Phi_p)T_0}+2h_1\cos(\Phi_1 T_0+\gamma_1)\Big]$$

$$+\delta_{j3}[\alpha_5A_1\,e^{i\omega_1 T_0}+\alpha_6A_2\,e^{i\omega_2 T_0}]+\text{NST}+\text{C.C.} \qquad (3.1, 3.2, 3.3)$$

where NST stands for terms that do not produce secular or small-divisor terms. Putting

$$\Omega_i=\omega_1+\varepsilon\sigma_3 \qquad \text{and} \qquad \Phi_1=\omega_2+\varepsilon\sigma_4 \qquad (3.4)$$

and eliminating the terms in equations (3.1)-(3.3) that produce secular terms in X_{j1} yields the following equations for A_j [using the polar form for $A_j = a_j(T_1)\, e^{i\beta_j(T_1)}$]:

$$a_j' = a_j\left(\xi_j - \frac{1}{8}a_j^2\omega_j^2\right) + \delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2 \sin\theta_1 + \frac{\alpha_3}{2\omega_1}a_3 \sin\theta_3 + \frac{f_1}{\omega_1}\sin\theta_3\right]$$
$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1 \sin\theta_1 + \frac{\alpha_4}{2\omega_2}a_3 \sin\theta_4 + \frac{h_1}{\omega_2}\sin\theta_5\right]$$
$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1 \sin\theta_2 - \frac{\alpha_6}{2\omega_1}a_2 \sin\theta_4\right] \qquad (3.5, 3.6, 3.7)$$

$$a_j\beta_j' = \delta_{j1}\left[-\frac{\alpha_1}{2\omega_1}a_2 \cos\theta_1 - \frac{\alpha_3}{2\omega_1}a_3 \cos\theta_2 - \frac{f_1}{\omega_1}\cos\theta_3\right]$$
$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1 \cos\theta_1 - \frac{\alpha_4}{2\omega_2}a_3 \cos\theta_4 - \frac{h_1}{\omega_2}\cos\theta_5\right]$$
$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1 \cos\theta_2 - \frac{\alpha_6}{2\omega_3}a_2 \cos\theta_4\right] \qquad (3.8, 3.9, 3.10)$$

where

$$\theta_1 = \sigma_1 T_1 + \beta_2 - \beta_1, \qquad \theta_2 = \sigma_2 T_1 + \beta_3 - \beta_1, \qquad \theta_3 = \sigma_3 T_1 - \beta_1 + \nu_1$$
$$\theta_4 = (\sigma_2 - \sigma_1)T_1 + \beta_3 - \beta_2, \qquad \theta_5 = \sigma_4 T_1 - \beta_2 + \gamma_1 \qquad (3.11)$$

and

$$\xi_1 = \frac{1}{2} - \sum_{n=1}^{N} P_n\bar{P}_n, \qquad \xi_2 = \frac{1}{2} - \sum_{m=1}^{M} P_m\bar{P}_m, \qquad \xi_3 = \frac{1}{2} \qquad (3.12)$$

Thus, the approximate solutions can be written in the form

$$X_j = a_j \cos(\omega_j t + \beta_j) + 2\delta_{j1}\sum_{n=2}^{N}|P_n|\Omega_n^{-1}\cos(\Omega_n t + \nu_n)$$
$$+2\delta_{j2}\sum_{m=2}^{M}|P_m|\Phi_m^{-1}\cos(\Phi_m t + \gamma_m) + o(\varepsilon) \qquad (3.13, 3.14, 3.15)$$

where a_j and β_j are defined by equations (3.5)-(3.10).

3.2. Steady-State Solutions

For the system of equations (3.5)-(3.10) to have stationary solutions, the following conditions must be satisfied:

$$a_1' = a_2' = a_3' = \theta_1' = \theta_2' = \theta_3' = \theta_4' = \theta_5' = 0 \qquad (3.16)$$

Hence, using the above conditions and substituting expressions (3.16) into equations (3.5)-(3.10) yields the steady-state equations

$$a_j\left(\xi_j-\frac{1}{8}a_j^2\omega_j^2\right)+\delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\sin\theta_1+\frac{\alpha_3}{2\omega_1}a_3\sin\theta_2+\frac{f_1}{\omega_1}\sin\theta_3\right]$$

$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\sin\theta_1+\frac{\alpha_4}{2\omega_2}a_3\sin\theta_4+\frac{h_1}{\omega_2}\sin\theta_5\right]$$

$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1\sin\theta_2-\frac{\alpha_6}{2\omega_3}a_2\cos\theta_4\right]=0 \qquad (3.17, 3.18, 3.19)$$

$$\delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\cos\theta_1+\frac{\alpha_3}{2\omega_1}a_3\cos\theta_2+\frac{f_1}{\omega_1}\cos\theta_3+a_1\sigma_3\right]$$

$$+\delta_{j2}\left[\frac{\alpha_2}{2\omega_2}a_1\cos\theta_1+\frac{\alpha_4}{2\omega_2}a_3\cos\theta_4+\frac{h_1}{\omega_2}\cos\theta_5+a_2(\sigma_3-\sigma_1)\right]$$

$$+\delta_{j3}\left[\frac{\alpha_5}{2\omega_3}a_1\cos\theta_2+\frac{\alpha_6}{2\omega_3}a_2\cos\theta_4+a_3(\sigma_3-\sigma_2)\right]=0 \qquad (3.20, 3.21, 3.22)$$

4. SUBHARMONIC RESONANCE

4.1. Approximate Solutions

In this case, we write $\Omega_q=3\omega_1+\varepsilon\sigma_3$ and $\Phi_q=3\omega_2+\varepsilon\sigma_4$. Eliminating the secular and the small-divisor terms from the solutions of equations (2.14)-(2.16) yields

$$a_j'=a_j\left(\xi_j-\frac{1}{8}a_j^2\omega_j^2\right)+\delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\sin\theta_1+\frac{\alpha_3}{2\omega_1}a_3\sin\theta_2+\omega_1a_1^2S_1\cos\theta_3\right]$$

$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_1}a_1\sin\theta_1+\frac{\alpha_4}{2\omega_2}a_3\sin\theta_4+\omega_2a_2^2S_2\cos\theta_5\right]$$

$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1\sin\theta_2-\frac{\alpha_6}{2\omega_3}a_3\sin\theta_4\right] \qquad (4.1, 4.2, 4.3)$$

$$a_j\beta_j'=\delta_{j1}\left[-\frac{\alpha_1}{2\omega_1}a_2\cos\theta_1-\frac{\alpha_3}{2\omega_1}a_3\cos\theta_2+\omega_1a_1^2S_1\sin\theta_3\right]$$

$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\cos\theta_1-\frac{4}{2\omega_2}a_3\cos\theta_4+\omega_2a_2^2S_2\sin\theta_5\right]$$

$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1\cos\theta_2-\frac{\alpha_6}{2\omega_3}a_2\cos\theta_4\right] \qquad (4.4, 4.5, 4.6)$$

where

$$\theta_3=\sigma_{3T_1}-3\beta_1+\nu_q, \qquad \theta_5=\sigma_4 T_1-3\beta_2+\gamma_p, \qquad S_1=\tfrac{1}{4}|P_q|, \qquad S_2=\tfrac{1}{4}|P_p| \tag{4.7}$$

Thus, the approximate solutions can be written in the form

$$X_j=a_j\cos(\omega_j t+\beta_j)+2\delta_{j1}\sum_{n=1}^{N}|P_n|\cos(\Omega_n t+\nu_n)$$

$$+2\delta_{j2}\sum_{m=1}^{M}|P_m|\cos(\Phi_m t+\gamma_m)+O(\varepsilon) \qquad (4.8, 4.9, 4.10)$$

where a_j and β_j are defined by equations (4.1)-(4.6).

4.2. Steady-State Solution

A stationary solution is not possible unless the conditions (3.16) are satisfied. Hence, the steady-state equations are

$$a_j\left(\xi_j-\frac{1}{8}a_j^2\omega_j^2\right)+\delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\sin\theta_1-\frac{\alpha_3}{2\omega_1}a_3\sin\theta_2+\omega_1 a_1^2 S_1\cos\theta_3\right]$$

$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\sin\theta_1+\frac{\alpha_4}{2\omega_2}a_3\sin\theta_4+\omega_2 a_2^2 S_2\cos\theta_5\right]$$

$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1\sin\theta_2-\frac{\alpha_6}{2\omega_2}a_2\sin\theta_4\right]=0 \qquad (4.11, 4.12, 4.13)$$

$$\delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\cos\theta_1+\frac{\alpha_3}{2\omega_1}a_3\cos\theta_2-\omega_1 a_1^2 S_1\sin\theta_3+\frac{1}{3}a_1\sigma_3\right]$$

$$+\delta_{j2}\left[\frac{\alpha_2}{2\omega_2}a_1\cos\theta_1+\frac{\alpha_4}{2\omega_2}a_3\cos\theta_4-\omega_2 a_2^2 S_2\sin\theta_5+a_2\left(\frac{1}{3}\sigma_3-\sigma_1\right)\right]$$

$$+\delta_{j3}\left[\frac{\alpha_5}{2\omega_3}a_1\cos\theta_2+\frac{\alpha_6}{2\omega_3}a_2\cos\theta_4+a_3\left(\frac{1}{3}\sigma_3-\sigma_2\right)\right]=0$$

$$(4.14, 4.15, 4.16)$$

5. SUPERHARMONIC RESONANCE

5.1. Approximate Solutions

In this case we write $3\Omega_p=\omega_1+\varepsilon\sigma_3$ and $3\Phi_p=\omega_2+\varepsilon\sigma_4$. Eliminating the secular and small-divisor terms from the solutions of equations

(2.14)-(2.16), yields

$$a_j' = a_j\left(\xi_j - \frac{1}{8}a_j^2\omega_j^2\right) + \delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\sin\theta_1 + \frac{\alpha_3}{2\omega_1}a_3\sin\theta_2 + \frac{K_1}{\omega_1}\cos\theta_3\right]$$
$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\sin\theta_1 + \frac{\alpha_4}{2\omega_2}a_3\sin\theta_4 + \frac{K_2}{\omega_2}\cos\theta_5\right]$$
$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_2}a_1\sin\theta_2 - \frac{\alpha_6}{2\omega_3}a_2\sin\theta_4\right] \qquad (5.1, 5.2, 5.3)$$

$$a_j\beta_j' = \delta_{j1}\left[-\frac{\alpha_1}{2\omega_1}a_2\cos\theta_1 - \frac{\alpha_3}{2\omega_1}a_3\cos\theta_2 + \frac{K_1}{\omega_1}\sin\theta_3\right]$$
$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\cos\theta_1 - \frac{\alpha_4}{2\omega_2}a_3\cos\theta_4 + \frac{K_2}{\omega_2}\sin\theta_5\right]$$
$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1\cos\theta_2 - \frac{\alpha_6}{2\omega_3}a_2\cos\theta_4\right] \qquad (5.4, 5.5, 5.6)$$

where

$$K_1 = \tfrac{1}{3}|P_p|^3, \qquad K_2 = \tfrac{1}{3}|P_q|^3, \qquad \theta_3 = \sigma_3 T_1 - \beta_1 + 3\nu_p, \qquad \theta_5 = \sigma_4 T_1 - \beta_2 + 3\gamma_q \qquad (5.7)$$

Thus, the approximate solutions are given by equations (4.8)-(4.10), where a_j and β_j are defined by equations (5.1)-(5.6).

5.2. Steady-State Solutions

Imposing the conditions (3.16) on equations (5.1)-(5.6) yields the following steady-state equations:

$$a_j\left(\xi_j - \frac{1}{8}a_j^2\omega_j^2\right) + \delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\sin\theta_1 + \frac{\alpha_3}{2\omega_1}a_3\sin\theta_2 + \frac{K_1}{\omega_1}\cos\theta_3\right]$$
$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\sin\theta_1 + \frac{\alpha_4}{2\omega_2}a_3\sin\theta_4 + \frac{K_2}{\omega_2}\cos\theta_5\right]$$
$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}\sin\theta_2 - \frac{\alpha_6}{2\omega_3}a_2\sin\theta_4\right] = 0 \qquad (5.8, 5.9, 5.10)$$

$$\delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\cos\theta_1 + \frac{\alpha_3}{2\omega_1}a_3\cos\theta_2 - \frac{K_1}{\omega_1}\sin\theta_3 + a_1\sigma_3\right]$$
$$+\delta_{j2}\left[\frac{\alpha_2}{2\omega_2}a_1\cos\theta_1 + \frac{\alpha_4}{2\omega_2}a_3\cos\theta_4 - \frac{K_2}{\omega_2}\sin\theta_5 + a_2(\sigma_3 - \sigma_1)\right]$$
$$+\delta_{j3}\left[\frac{\alpha_5}{2\omega_3}a_1\cos\theta_2 + \frac{\alpha_6}{2\omega_3}a_2\cos\theta_4 + a_3(\sigma_3 - \sigma_2)\right] = 0$$
$$(5.11, 5.12, 5.13)$$

6. SIMULTANEOUS SUB/SUPERHARMONIC RESONANCES

6.1. Approximate Solutions

In this case, we take

$$3\Omega_p = \omega_1 + \varepsilon\sigma_3, \qquad \Omega_q = 3\omega_1 + \varepsilon\sigma_4$$

$$3\Phi_p = \omega_2 + \varepsilon\sigma_5, \qquad \Phi_q = 3\omega_2 + \varepsilon\sigma_6$$

Eliminating the terms in equations (2.14)-(2.16) that produce secular terms yields

$$a_j' = a_j\left(\xi_j - \frac{1}{8}a_j^2\omega_j^2\right) + \delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\sin\theta_1 + \frac{\alpha_3}{2\omega_1}a_3\sin\theta_2 + \frac{K_1}{\omega_1}\cos\theta_3 + \omega_1 a_1^2 S_1\cos\theta_4\right]$$
$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\sin\theta_1 + \frac{\alpha_4}{2\omega_2}a_3\sin\theta_5 + \frac{K_2}{\omega_2}\cos\theta_6 + \omega_2 a_2^2 S_2\cos\theta_7\right]$$
$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1\sin\theta_2 - \frac{\alpha_6}{2\omega_3}a_2\sin\theta_5\right] \qquad (6.1, 6.2, 6.3)$$

$$a_j\beta_j' = \delta_{j1}\left[-\frac{\alpha_1}{2\omega_j}a_2\cos\theta_1 - \frac{\alpha_3}{2\omega_1}a_3\cos\theta_2 + \frac{K_1}{\omega_1}\sin\theta_3 + \omega_1 a_1^2 S_1\sin\theta_4\right]$$
$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\cos\theta_1 - \frac{\alpha_4}{2\omega_2}a_3\cos\theta_5 + \frac{K_2}{\omega_2}\sin\theta_6 + \omega_2 a_2^2 S_2\sin\theta_7\right]$$
$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1\cos\theta_2 - \frac{\alpha_6}{2\omega_3}a_2\cos\theta_5\right] \qquad (6.4, 6.5, 6.6)$$

where

$$\theta_3 = \sigma_3 T_1 - \beta_1 + 3\nu_p, \qquad \theta_4 = \sigma_4 T_1 - 3\beta_2 + \nu_q,$$
$$\theta_5 = (\sigma_2 - \sigma_1)T_1 + \beta_3 - \beta_2 \qquad \theta_6 = \sigma_5 T_1 - \beta_2 + 3\gamma_{p1}, \qquad (6.7)$$
$$\theta_7 = \sigma_6 T_1 - 3\beta_2 + \gamma_{q1}$$

Thus, the approximate solutions are given by equations (4.8)-(4.10), where a_j and β_j are defined by equations (6.1)-(6.6).

6.2. Steady-State Solutions

A stationary solution is not possible unless the conditions (3.16) are satisfied. Hence, the steady-state equations are

$$a_j\left(\xi_j-\frac{1}{8}a_j^2\omega_j^2\right)+\delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\sin\theta_1+\frac{\alpha_3}{2\omega_1}a_3\sin\theta_2\right.$$
$$\left.+\frac{K_1}{\omega_1}\cos\theta_3+\omega_1 a_1^2 S_1\cos\theta_4\right]$$
$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\sin\theta_1+\frac{\alpha_4}{2\omega_2}a_3\sin\theta_5+\frac{K_2}{\omega_2}\cos\theta_6\right.$$
$$\left.+\omega_2 a_2^2 S_2\sin\theta_7\right]$$
$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1\sin\theta_2-\frac{\alpha_6}{2\omega_3}a_2\sin\theta_5\right]=0 \qquad (6.8, 6.9, 6.10)$$

$$\delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\cos\theta_1+\frac{\alpha_3}{2\omega_1}a_3\cos\theta_2-\frac{K_1}{\omega_1}\sin\theta_3-\omega_1 a_1^2 S_1\sin\theta_4+a_1\sigma_3\right]$$
$$+\delta_{j2}\left[\frac{\alpha_2}{2\omega_2}a_1\cos\theta_1+\frac{\alpha_4}{2\omega_2}a_3\cos\theta_5-\frac{K_2}{\omega_2}\sin\theta_6\right.$$
$$\left.-\omega_2 a_2^2 S_2\sin\theta_7+a_2(\sigma_2-\sigma_1)\right]$$
$$+\delta_{j3}\left[\frac{\alpha_5}{2\omega_3}a_1\cos\theta_2+\frac{\alpha_6}{2\omega_3}a_2\cos\theta_5+a_3(\sigma_3-\sigma_1)\right]=0$$
$$(6.11, 6.12, 6.13)$$

7. COMBINATION RESONANCE

7.1. Approximate Solutions

In this case, we take $\Omega_n+\Omega_p=2\omega_1+\varepsilon\sigma_3$ and $\Phi_m+\Phi_p=2\omega_2+\varepsilon\sigma_4$. Eliminating the terms in equations (2.14)-(2.16) that produce secular terms yields

$$a_j' = a_j\left(\xi_j - \frac{1}{8}a_j^2\omega_j^2\right)$$
$$+\delta_{j1}\left[\frac{1}{2\omega_1}a_2\sin\theta_1 + \frac{\alpha_3}{2\omega_1}a_3\sin\theta_2 - \frac{1}{2}a_1d_1\cos\theta_3\right]$$
$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\sin\theta_1 + \frac{\alpha_4}{2\omega_2}a_3\sin\theta_4 - \frac{1}{2}a_2d_2\cos\theta_4\right]$$
$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1\sin\theta_2 - \frac{\alpha_6}{2\omega_3}a_2\sin\theta_4\right] \qquad (7.1, 7.2, 7.3)$$

$$a_j\beta_j' = \delta_{j1}\left[-\frac{\alpha_1}{2\omega_1}a_2\cos\theta_1 - \frac{\alpha_3}{2\omega_1}a_3\cos\theta_2 - \frac{1}{2}a_1d_1\sin\theta_3\right]$$
$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\cos\theta_1 - \frac{\alpha_4}{2\omega_2}a_3\cos\theta_4 - \frac{1}{2}a_2d_2\sin\theta_5\sin\theta_5\right]$$
$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1\cos\theta_2 - \frac{\alpha_6}{2\omega_3}a_2\cos\theta_4\right] \qquad (7.4, 7.5, 7.6)$$

where

$$d_1 = |P_nP_p|, \qquad d_2 = |P_mP_p|$$
$$\theta_3 = \sigma_3T_1 - 2\beta_1 + \nu_n + \nu_p, \qquad \theta_5 = \sigma_4T_1 - 2\beta_2 + \gamma_m + \gamma_p \qquad (7.7)$$

Thus, the approximate solutions are given by equations (4.8)-(4.10), where a_j and β_j are defined by equations (7.1)-(7.6).

7.2. Steady-State Solutions

Imposing the conditions (3.16) on equations (7.1)-(7.6) yields the following steady-state equations:

$$a_j\left(\xi_j - \frac{1}{8}a_j^2\omega_j^2\right) + \delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\sin\theta_1 + \frac{\alpha_3}{2\omega_1}a_3\sin\theta_2 - \frac{1}{2}a_1d_1\cos\theta_3\right]$$
$$+\delta_{j2}\left[-\frac{\alpha_2}{2\omega_2}a_1\sin\theta_1 + \frac{\alpha_4}{2\omega_2}a_3\sin\theta_4 - \frac{1}{2}a_2d_2\cos\theta_5\right]$$
$$+\delta_{j3}\left[-\frac{\alpha_5}{2\omega_3}a_1\sin\theta_2 - \frac{\alpha_6}{2\omega_3}a_2\sin\theta_4\right] = 0 \qquad (7.8, 7.9, 7.10)$$

$$\delta_{j1}\left[\frac{\alpha_1}{2\omega_1}a_2\cos\theta_1+\frac{\alpha_3}{2\omega_1}a_3\cos\theta_2+\frac{1}{2}a_1d_1\sin\theta_3+\frac{1}{2}a_1\sigma_3\right]$$

$$+\delta_{j2}\left[\frac{\alpha_2}{2\omega_2}a_1\cos\theta_1+\frac{\alpha_4}{2\omega_2}a_3\cos\theta_4\right.$$

$$\left.+\frac{1}{2}a_2d_2\sin\theta_5+a_2\left(\frac{1}{2}\sigma_3-\sigma_1\right)\right]$$

$$+\delta_{j3}\left[\frac{\alpha_5}{2\omega_3}a_1\cos\theta_2+\frac{\alpha_6}{2\omega_3}a_2\cos\theta_4\right.$$

$$\left.-a_3\left(\frac{1}{2}\sigma_3-\sigma_2\right)\right]=0 \qquad (7.11, 7.12, 7.13)$$

8. NUMERICAL RESULTS

From equations (3.17)-(3.22), (4.11)-(4.16), (5.8)-(5.13), (6.8)-(6.13), and (7.8)-(7.13) we can obtain the corresponding frequency response equations, which are nonlinear coupled equations and can be solved numerically by using the Newton-Raphson method. Choosing suitable initial values, for example, $X_0=3$, $Y_0=3$, and $Z_0=3$, we obtain the results in Figures 1-19. Figures 1-5 show the corresponding variation, in the case of harmonic resonance, of the parameters ξ_1, σ_1, and σ_2, and when the coupling terms $\alpha_1=\alpha_5=0$, the three modes are decreasing in magnitude. Figures 6-9 show the variation, in the case of subharmonic resonance, of the parameters σ_1 and σ_2, the three modes are decreasing in magnitude with respect to Figure 6. Figures 9-12 correspond to the variation, in the case of superharmonic resonance, of the parameters F_p and σ_2, and the three modes are decreasing in magnitude with respect to Figures 9, 10, and 12. When σ_1 increases, the first and second modes decrease in magnitude and the third mode increases in magnitudes (Figure 11). Figures 13-15 show the variation of the parameters σ_1 and σ_2 in the case of simultaneous sub/superharmonic resonances. The three modes have decreasing magnitudes with respect to Figure 13. Figures 16-19 correspond to variations, in the case of combination resonance, of the parameters η_1, σ_1, σ_2; the three modes are decreasing in magnitude with respect to Figure 16. For all the figures, we take $\omega_1=1$, $\omega_2=1$, and $\omega_3=1$.

9. STABILITY ANALYSIS

To determine the stability of the steady-state solutions, one lets

$$a_j=a_{j0}+a_{j1} \quad \text{and} \quad \theta_j=\theta_{j0}+\theta_{j1}, \quad j=1,2,3,4,5 \qquad (9.1)$$

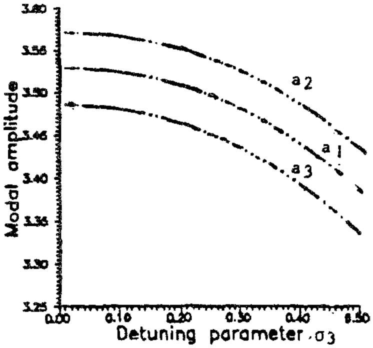

Fig. 1. $r_1 = 0.2$, $h_1 = 0.4$, $\xi_1 = 0.5$, $\xi_2 = 0.5$, $\sigma_1 = \sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

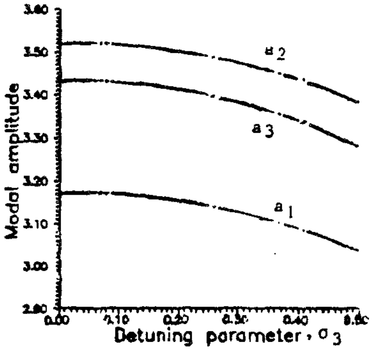

Fig. 2. $f_1 = 0.2$, $h_1 = 0.4$, $\xi_1 = 0.1$, $\xi_2 = 0.5$, $\sigma_1 = \sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

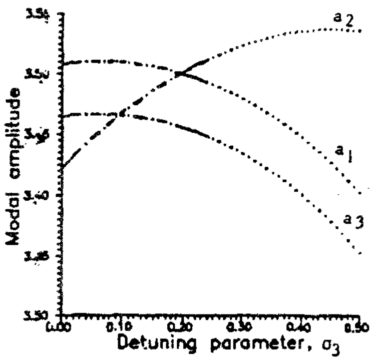

Fig. 3. $f_1 = 0.2$, $h_1 = 0.4$, $\xi_1 = 0.5$, $\xi_2 = 0.5$, $\sigma_1 = 0.6$, $\sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

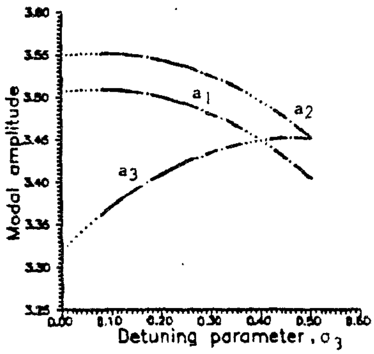

Fig. 4. $f_1 = 0.2$, $h_1 = 0.4$, $\xi_1 = 0.5$, $\xi_2 = 0.5$, $\sigma_1 = 0$, $\sigma_2 = 0.6$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

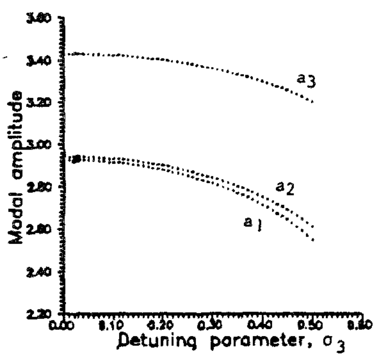

Fig. 5. $f_1 = 0.2$, $h_1 = 0.4$, $\xi_1 = 0.5$, $\xi_2 = 0.5$, $\sigma_1 = \sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

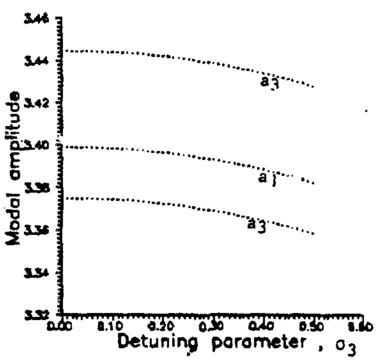

Fig. 6. $F_q = 1.3$, $H_q = 1.6$, $\xi_1 = 0.441$, $\xi_2 = 0.41$, $\sigma_1 = \sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

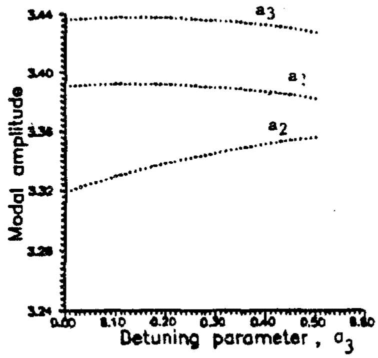

Fig. 7. $F_q = 1.3$, $H_p = 1.6$, $\xi_1 = 0.441$, $\xi_2 = 0.41$, $\sigma_1 = 0.35$, $\sigma_2 = 0$, $\alpha_j = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

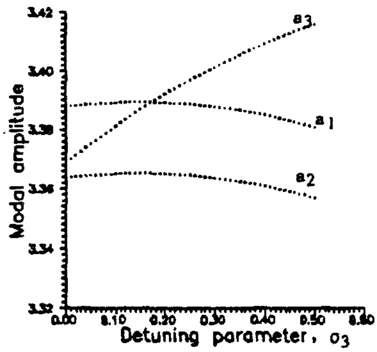

Fig. 8. $F_q = 1.3$, $H_p = 1.6$, $\xi_1 = 0.441$, $\xi_2 = 0.41$, $\sigma_1 = 0$, $\sigma_2 = 0.4$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

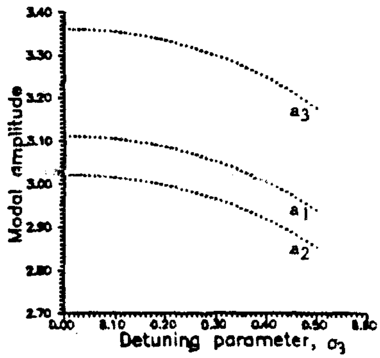

Fig. 9. $F_p = 3$, $H_q = 3.5$, $\xi_1 = 0.184$, $\xi_2 = 0.07$, $\sigma_1 = \sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

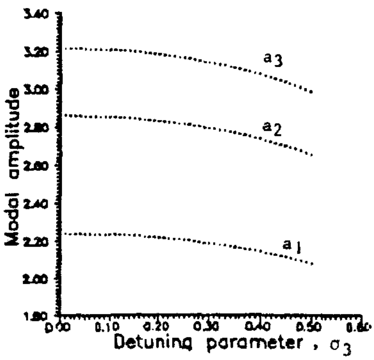

Fig. 10. $F_p = 6$, $H_p = 3.5$, $\xi_1 = -0.77$, $\xi_2 = 0.07$, $\sigma_1 = \sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

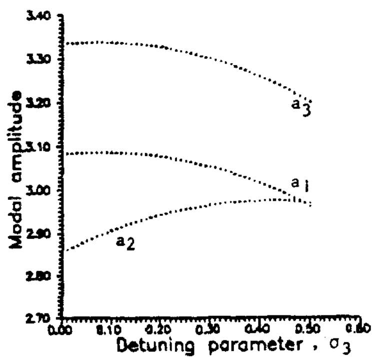

Fig. 11. $F_p = 3$, $H_p = 3.5$, $\xi_1 = 0.184$, $\xi_2 = 0.07$, $\sigma_1 = 0.6$, $\sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

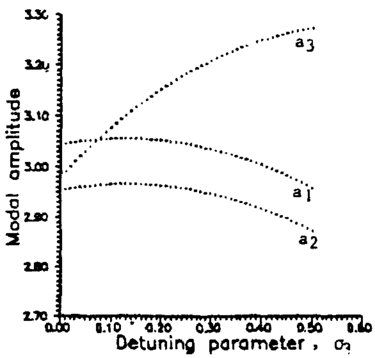

Fig. 12. $F_p = 3$, $H_p = 3.5$, $\xi_1 = 0.184$, $\xi_2 = 0.07$, $\sigma_1 = 0$, $\sigma_2 = 0.8$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

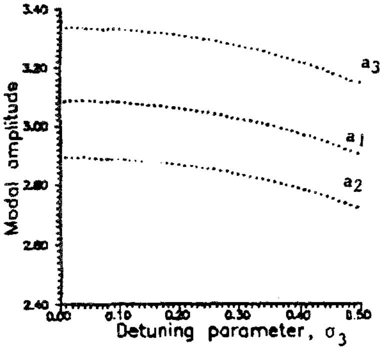

Fig. 13. $F_p = F_q = 3$, $H_p = H_q = 4$, $\xi_1 = 0.184$, $\xi_2 = -0.063$, $\sigma_1 = \sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{34} = \theta_{25} = \theta_{16} = \theta_{56} = \theta_{57} = \theta_{67} = 0$.

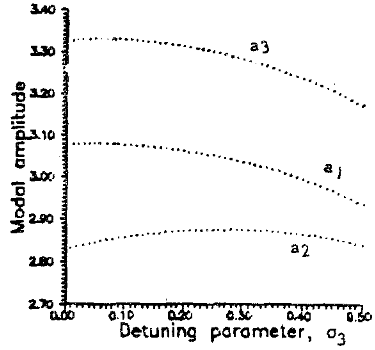

Fig. 14. $F_p = F_q = 3$, $H_p = H_q = 4$, $\xi_1 = 0.164$, $\xi_2 = -0.063$, $\sigma_1 = 0.4$, $\sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{34} = \theta_{15} = \theta_{25} = \theta_{16} = \theta_{56} = \theta_{57} = \theta_{56} = \theta_{67} = 0$.

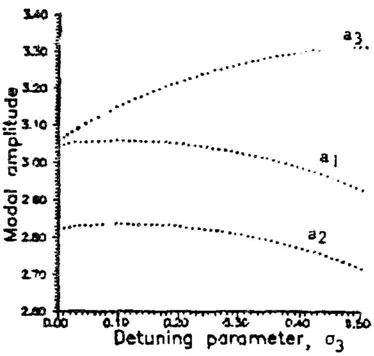

Fig. 15. $F_p = F_q = 3$, $H_p = H_q = 4$, $\xi_1 = 0.184$, $\xi_2 = -0.063$, $\sigma_1 = 0$, $\sigma_2 = 0.7$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{34} = \theta_{15} = \theta_{25} = \theta_{16} = \theta_{56} = \theta_{57} = \theta_{56} = \theta_{67} = 0$.

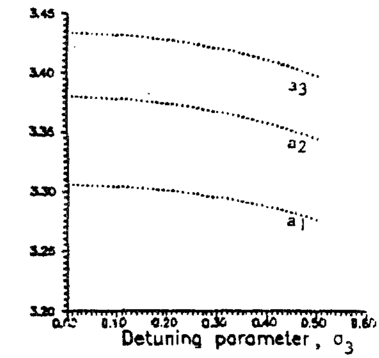

Fig. 16. $d_1 = 0.3$, $d_2 = 0.5$, $\xi_1 = 0.1$, $\xi_2 = -0.4$, $\sigma_1 = \sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{15} = \theta_{45} = 0$.

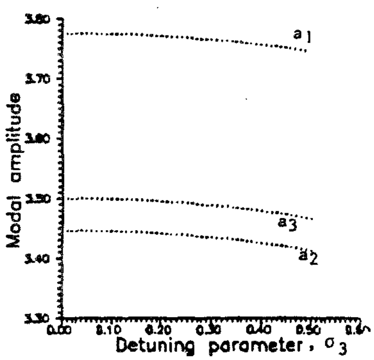

Fig. 17. $d_1 = 0.3$, $d_2 = 0.5$, $\xi_1 = 0.7$, $\xi_2 = 0.4$, $\sigma_1 = \sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

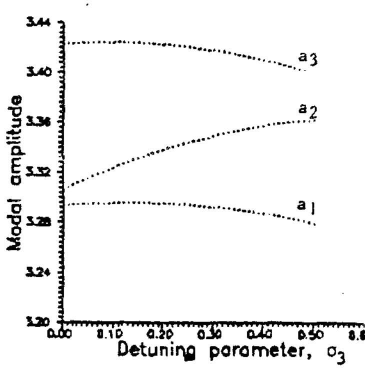

Fig. 18. $d_1 = 0.3$, $d_2 = 0.5$, $\xi_1 = 0.1$, $\xi_2 = 0.4$, $\sigma_1 = 0.4$, $\sigma_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

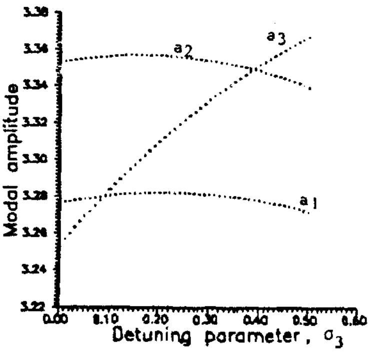

Fig. 19. $d_1 = 0.3$, $d_2 = 0.5$, $\xi_1 = 0.1$, $\xi_2 = 0.4$, $\sigma_1 = 0$, $\sigma_2 = 0.6$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{14} = \theta_{24} = \theta_{15} = \theta_{45} = 0$.

where the a_{j0} and θ_{j0} are the steady-state values and a_{j1} and θ_{j1} are infinitesimal time-dependent perturbations. Now we study the stability corresponding to the harmonic case.

Substituting (9.1) into equations (3.5)-(3.10), using equations (3.17)-(3.22), and linearizing the resulting equations, one obtains

$$\begin{aligned}
a'_{j1} = {} & \left(\xi_j - \frac{3}{8} a_{j0}^2 \omega_j^2\right) a_{j1} + \delta_{j1}\left[\frac{\alpha_1}{2\omega_1}(\sin\theta_{10}) a_{21} + \frac{\alpha_3}{2\omega_1}(\sin\theta_{20}) a_{31}\right. \\
& \left. + \frac{\alpha_1}{2\omega_1}(a_{20}\cos\theta_{10})\theta_{11} + \frac{\alpha_3}{2\omega_1}(a_{30}\cos\theta_{20})\theta_{21} + \frac{f_1}{\omega_1}(\cos\theta_{30})\theta_{31}\right] \\
& + \delta_{j2}\left\{-\frac{\alpha_2}{2\omega_2}(\sin\theta_{10}) a_{11} + \frac{\alpha_4}{2\omega_2}(\sin\theta_{40}) a_{31}\right. \\
& - \left[\frac{\alpha_2}{2\omega_2}(a_{10}\cos\theta_{10}) + \frac{\alpha_4}{2\omega_2}(a_{30}\cos\theta_{40}) - \frac{h_1}{\omega_2}(\cos\theta_{50})\right]\theta_{11} \\
& \left. + \frac{\alpha_4}{2\omega_2}(a_{30}\cos\theta_{40})\theta_{21} + \frac{h_1}{\omega_2}(\cos\theta_{50})\theta_{31}\right\} \\
& + \delta_{j3}\left\{-\frac{\alpha_5}{2\omega_3}(\sin\theta_{20}) a_{11} - \frac{\alpha_6}{2\omega_3}(\sin\theta_{40}) a_{21}\right. \\
& + \frac{\alpha_6}{2\omega_3}(a_{20}\cos\theta_{40})\theta_{11} \\
& \left. - \left[\frac{\alpha_5}{2\omega_3}(a_{10}\cos\theta_{20}) + \frac{\alpha_6}{2\omega_3}(a_{20}\cos\theta_{40})\right]\theta_{21}\right\} \qquad (9.2, 9.3, 9.4)
\end{aligned}$$

$$
\begin{aligned}
\theta'_{j1} = \sigma_{j1}\Bigg\{ & \left[\frac{\sigma_1}{a_{10}} - \frac{\alpha_2}{2\omega_2}\left(\frac{1}{a_{20}}\cos\theta_{10}\right) - \frac{h_1}{\omega_1}\left(\frac{1}{a_{10}a_{20}}\cos\theta_{50}\right)\right] a_{11} \\
& + \left[\frac{\sigma_1}{a_{20}} + \frac{\alpha_1}{2\omega_1}\left(\frac{1}{a_{10}}\cos\theta_{10}\right) + \frac{f_1}{\omega_1}\left(\frac{1}{a_{10}a_{20}}\cos\theta_{30}\right)\right] a_{21} \\
& + \left[\frac{\sigma_1}{a_{30}} + \frac{\alpha_3}{2\omega_1}\left(\frac{1}{a_{10}}\cos\theta_{20}\right) - \frac{\alpha_4}{2\omega_2}\left(\frac{1}{a_{20}}\cos\theta_{40}\right)\right] a_{31} \\
& + \left[\frac{\alpha_2}{2\omega_2}\left(\frac{a_{10}}{a_{20}}\sin\theta_{10}\right) - \frac{\alpha_4}{2\omega_2}\left(\frac{a_{30}}{a_{20}}\sin\theta_{40}\right)\right. \\
& \left. - \frac{h_1}{\omega_2}\left(\frac{1}{a_{20}}\sin\theta_{50}\right) - \frac{\alpha_1}{2\omega_1}\left(\frac{a_{20}}{a_{10}}\sin\theta_{10}\right)\right] \theta_{11} \\
& + \left[\frac{\alpha_4}{2\omega_2}\left(\frac{a_{30}}{a_{20}}\sin\theta_{40}\right) - \frac{\alpha_3}{2\omega_1}\left(\frac{a_{30}}{a_{10}}\sin\theta_{20}\right)\right] \theta_{21} \\
& + \left[\frac{h_1}{\omega_2}\left(\frac{1}{a_{20}}\sin\theta_{50}\right) - \frac{f_1}{\omega_1}\left(\frac{1}{a_{10}}\sin\theta_{30}\right)\right] \theta_{31} \Bigg\} \\
& + \sigma_{j2}\Bigg\{ \left[\frac{\sigma_2}{a_{10}} - \frac{\alpha_5}{2\omega_3}\left(\frac{1}{a_{30}}\cos\theta_{20}\right)\right] a_{11} \\
& + \left[\frac{\alpha_1}{2\omega_1}\left(\frac{1}{a_{10}}\cos\theta_{10}\right) + \frac{f_1}{\omega_1}\left(\frac{1}{a_{10}a_{20}}\cos\theta_{30}\right)\right. \\
& \left. - \frac{\alpha_6}{2\omega_2}\left(\frac{1}{a_{30}}\cos\theta_{40}\right)\right] a_{21} + \frac{\alpha_3}{2\omega_1}\left(\frac{1}{a_{10}}\cos\theta_{20}\right) a_{31} \\
& - \left[\frac{\alpha_6}{2\omega_3}\left(\frac{a_{20}}{a_{30}}\sin\theta_{40}\right) + \frac{\alpha_1}{2\omega_1}\left(\frac{a_{20}}{a_{10}}\sin\theta_{11}\right)\right] \theta_{11} \\
& + \left[\frac{\alpha_5}{2\omega_3}(a_{10}\sin\theta_{20}) + \frac{\alpha_6}{2\omega_3}\left(\frac{a_{20}}{a_{30}}\sin\theta_{40}\right)\right. \\
& \left. - \frac{\alpha_3}{2\omega_1}\left(\frac{a_{30}}{a_{10}}\sin\theta_{20}\right)\right] \theta_{21} - \frac{f_1}{\omega_1}\left(\frac{1}{a_{10}}\sin\theta_{30}\right) \theta_{31} \Bigg\} \\
& + \delta_{j3}\Bigg\{ \frac{\sigma_3}{a_{10}} a_{11} + \left[\frac{\alpha_1}{2\omega_1}\left(\frac{1}{a_{10}}\cos\theta_{10}\right) + \frac{K_1}{\omega_1}\left(\frac{1}{a_{10}a_{20}}\cos\theta_{30}\right)\right] a_{21} \\
& + \frac{\alpha_3}{2\omega_1}\left(\frac{1}{a_{10}}\cos\theta_{20}\right) a_{31} - \frac{\alpha_1}{2\omega_1}\left(\frac{a_{20}}{a_{10}}\sin\theta_{10}\right) \theta_{11} \\
& - \frac{\alpha_3}{2\omega_1}\left(\frac{a_{30}}{a_{10}}\sin\theta_{20}\right) \theta_{21} - \frac{K_1}{\omega_1}\left(\frac{1}{a_{10}}\sin\theta_{30}\right) \theta_{31} \Bigg\} \qquad (9.5, 9.6, 9.7)
\end{aligned}
$$

Put $a_{j1} = C_j e^{\lambda T_1}$ and $\theta_{j1} = C_{j+3} e^{\lambda T_1}$ into equations (9.2)-(9.7). The solution is stable if and only if the real part of each of the eigenvalues of the coefficient of the matrix is less than or equal to zero. Following the same analysis, we can study the problem of stability for cases of other resonances. The stable solutions are represented by solid lines and the unstable solutions by broken lines on the frequency response curves (Figures 1-19). For harmonic solutions, the stable regions are discontinuous. For other resonances the solutions are not stable.

10. SUMMARY AND CONCLUSION

In this paper we have presented an analysis of the response of a self-excited three-degree-of-freedom systems in which the first two modes are excited externally to multifrequency excitation and the third mode is excited through coupling. The method of multiple scales is used to determine six first-order ordinary differential equations that describe the modulation of the amplitudes and phases of both types of modes caused by the resonances. Steady-state solutions are obtained and their stabilities are investigated. Numerical analyses are carried for different resonances, using the Newton-Raphson method, for suitable initial values. Numerical results are presented in Figures 1-19. The results show the effects of the variation of the parameters on the amplitudes.

REFERENCES

Elnaggar, A. M. (1976). Determination de solutions harmoniques et sous-harmoniques de l'equation $\ddot{X} + K_1X + K_2(\cos t)X^3 = 0$, Thesis, Strasbourg, France.

Elnaggar, A. M., and El-Bassiouny, A. F. (1991). Harmonic, subharmonic, superharmonic, simultaneous sub/superharmonic and combination resonances of self-excited two coupled second order systems to multi-frequency excitation, in *First ICEMP*, Cairo, Egypt.

Haag, J. (1962). *Oscillatory Motion*, Wodsworth Publishing Company, Belmont, California.

Kononenko, V. O., and Kovalchak, P. S. (1973). *Soviet Applied Mechanics*, **9**, 699.

Krylov, N. N., and Bogoliubov, N. N. (1947). *Introduction to Non-linear Mechanics*, Princeton University Press, Princeton, New Jersey.

Lindsay, W. C. (1972). *Synchronization Systems in Communication and Control*, Prentice-Hall, Englewood Cliffs, New Jersey.

Minorsky, N. (1962). *Non-linear Oscillations*, Van Nostrand, Princeton, New Jersey.

Nayfeh, A. H. (1983). *Journal of Sound and Vibration*, **20**(2), 237-244.

Nayfeh, A. H., and Mook, D. T. (1979). *Non-linear Oscillations*, Wiley-Interscience, New York.

Nayfeh, A. H., and Zavodney, L. D. (1986). *Journal of Sound and Vibration*, **107**(2), 329-350.

Pavlidis, T. (1962). *Biological Oscillators, Their Mathematical Analysis*, Academic Press, New York.

Tso, W. K., and Asmis, K. G. (1974). *International Journal of Non-Linear Mechanics*, **9**, 264-272.

Vasilenko, N. N. (1969). *Soviet Applied Mechanics*, **3**, 47.